

# On the Construction of Isospectral Vectorial Sturm-Liouville Differential Equations

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March 25, 1998

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## Abstract

We extend the idea of Jodeit and Levitan [3] for constructing isospectral problems of the classical scalar Sturm-Liouville differential equations to the vectorial Sturm-Liouville differential equations. Some interesting relations are found.

**Keywords and phrases.** vectorial Sturm-Liouville differential equation, matrix differential equations, isospectral problem

**AMS(MOS) subject classifications.** 34A30, 34B25.

Abbreviated title: Isospectral Problems

# 1 Introduction

The purpose of this article is to study the problem of constructing  $N$ -dimensional,  $N \geq 2$ , vectorial Sturm-Liouville differential equations subject to certain boundary conditions, which are isospectral to a given one.

Those  $N$ -dimensional vectorial Sturm-Liouville eigenvalue problems which are considered in this paper, are of the following form:

$$(1) \quad -\phi''(x) + P(x)\phi(x) = \lambda\phi(x), \quad B\phi'(0) + A\phi(0) = \mathcal{B}\phi'(\pi) + \mathcal{A}\phi(\pi) = \mathbf{0},$$

where  $0 \leq x \leq \pi$ ,  $\phi(x)$  is an  $\mathbf{R}^N$ -valued function,  $P(x)$  is a continuous  $N \times N$  symmetric matrix-valued function,  $A, B, \mathcal{A}, \mathcal{B}$  are  $N \times N$  matrices which satisfy the following conditions:

$$(2) \quad BA^* = AB^*, \mathcal{B}\mathcal{A}^* = \mathcal{A}\mathcal{B}^*, \quad \text{rank}[A, B] = \text{rank}[\mathcal{A}, \mathcal{B}] = N,$$

where  $A^*$  is the transpose matrix of  $A$ ,  $[A, B]$  denotes the  $N \times 2N$  matrix whose first  $N \times N$  block is  $A$ , and the second  $N \times N$  block is  $B$ . We shall use the tuple  $(P, A, B, \mathcal{A}, \mathcal{B})$  to denote the eigenvalue problem (1). Note that the conditions in (2) ensure the problem (1) a selfadjoint eigenvalue problem, and its eigenvalues can be determined by the variational principle. Counting multiplicities of the eigenvalues, we arrange the eigenvalues of (1) in an ascending sequence

$$(3) \quad \mu_0 \leq \mu_1 \leq \mu_2 \leq \cdots.$$

This sequence shall be denote by  $\Sigma(P, A, B, \mathcal{A}, \mathcal{B})$ , and is called the *sequence of eigenvalues* of (1). Note that the multiplicity of each eigenvalue of (1) is at most  $N$ . For convenience, we shall use  $\sigma(P, A, B, \mathcal{A}, \mathcal{B})$  to denote the *set of eigenvalues* of (1), and arrange its elements in ascending order as

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots,$$

and use  $m_k$  denote the multiplicity of  $\lambda_k$  in the sequence (3). Given two  $N$ -dimensional selfadjoint vectorial Sturm-Liouville eigenvalue problem  $(P, A, B, \mathcal{A}, \mathcal{B})$  and  $(\tilde{P}, \tilde{A}, \tilde{B}, \tilde{\mathcal{A}}, \tilde{\mathcal{B}})$  over  $0 \leq x \leq \pi$ . If  $\Sigma(P, A, B, \mathcal{A}, \mathcal{B}) = \Sigma(\tilde{P}, \tilde{A}, \tilde{B}, \tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ , we call these two eigenvalue problems *isospectral problems*, or  $(P, A, B, \mathcal{A}, \mathcal{B})$  is *isospectral* to  $(\tilde{P}, \tilde{A}, \tilde{B}, \tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ . For scalar Sturm-Liouville equations, i.e., (1) with  $N = 1$ , isospectral problems have been studied by many mathematicians, notably, G. Borg [1], I. M. Gel'fand, B. M. Levitan and their associates, and the structure of the set of isospectral problems for scalar Sturm-Liouville equations is well-presented in the book of J. Pöschel and E. Trubowitz [5], and in the works of E. Trubowitz. But for vectorial Sturm-Liouville equations, i.e., (1) with  $N \geq 2$ , methods for constructing isospectral problems and the structure of

isospectral problems are not well-understood. Motivated by a recent work of Jodeit and Levitan [3], in this paper we present a method for constructing an  $N$ -dimensional vectorial Sturm-Liouville eigenvalue problem  $(\tilde{P}, \tilde{A}, \tilde{B}, \tilde{\mathcal{A}}, \tilde{\mathcal{B}})$  over  $0 \leq x \leq \pi$ , which is isospectral to the given  $N$ -dimensional vectorial Sturm-Liouville eigenvalue problem  $(P, A, B, \mathcal{A}, \mathcal{B})$  over  $0 \leq x \leq \pi$ . For simplicity we shall assume the completeness of the eigenfunctions of the given eigenvalue problem  $(P, A, B, \mathcal{A}, \mathcal{B})$  subject to the given boundary conditions shown in (1).

This paper is organized as follows. In section 2 we present some preliminary results related to vectorial Sturm-Liouville equations, and a matrix wave equation (see (18)) which is constructed from the given eigenvalue problem  $(P, A, B, \mathcal{A}, \mathcal{B})$ . In section 3, using matrix wave equation introduced in section 2, we construct eigenvalue problems which are isospectral to  $(P, A, B, \mathcal{A}, \mathcal{B})$ . In section 4 we present two examples using our construction method. As shown in one of our examples, even for a simple case such as  $(P, I, 0, I, 0)$ , where  $P(x)$  is a constant two by two diagonal matrix,  $I$  is the two by two identity matrix, an isospectral problem  $(Q, I, 0, I, 0)$  can be found where  $Q(x)$  is not simultaneously diagonalizable. The isospectral problem for vectorial Sturm-Liouville equations is much more complicated than its scalar counterparts.

## 2 Preliminary

To study the eigenvalue problem (1) we introduce the following matrix differential equation

$$(4) \quad -Y'' + P(x)Y = \lambda Y, \quad Y(0) = B^*, \quad Y'(0) = -A^*.$$

Let  $Y(x; \lambda)$  denote the  $N \times N$  matrix-valued solution of the initial value problem (4). We have ( see [2] ),

$$Y(x; \lambda) = \mathcal{C}(x; \mu) + \int_0^x \tilde{K}(x, t) \mathcal{C}(t; \mu) dt$$

where  $\mu^2 = \lambda$ ,  $\mathcal{C}(x; \mu) = \cos(\mu x)B^* - \mu^{-1} \sin(\mu x)A^*$ , and  $\tilde{K}(x, t)$  is as that described in [2, **Lemma 2.1**]. Define the following matrix-valued function

$$(5) \quad W(\lambda) = \mathcal{B}Y'(\pi; \lambda) + \mathcal{A}Y(\pi; \lambda).$$

Then  $\lambda_* \in \sigma(P, A, B, \mathcal{A}, \mathcal{B})$  if and only if  $W(\lambda_*)$  is a singular matrix. It follows from the variational principle that the set of the zeros of the equation

$$(6) \quad \det W(\lambda) = 0$$

is bounded below. Denote the distinct zeros of (6) in stricting ascending order as

$$(7) \quad \lambda_0 < \lambda_1 < \lambda_2 < \cdots.$$

Then the multiplicity  $m_k$  of  $\lambda_k$  in the sequence of eigenvalues  $\Sigma(P, A, B, \mathcal{A}, \mathcal{B})$  of (1) is equal to the dimension of the null space  $\text{Null}(W(\lambda_k))$  of  $W(\lambda_k)$ . If  $\mathbf{v}$  is a nonzero element in  $\text{Null}(W(\lambda_k))$ , then the vector-valued function

$$z(x) = Y(x; \lambda_k) \mathbf{v}$$

is an eigenfunction of (1) corresponding to the eigenvalue  $\lambda_k$ . In the following, for  $\mathbf{v}$  and  $\mathbf{w}$  in  $\mathbf{R}^N$ , the notation  $\langle \mathbf{v}, \mathbf{w} \rangle$  is used to denote the inner product of two elements  $\mathbf{v}$  and  $\mathbf{w}$  in  $\mathbf{R}^N$ . We shall need the following result.

**Lemma 2.1** *For each  $k \geq 0$ , in the null space  $\text{Null}(W(\lambda_k))$  there are exactly  $m_k$  linearly independent constant vectors  $\theta_l(k)$ ,  $1 \leq l \leq m_k$ , such that the vector-valued functions*

$$Y(x; \lambda_k) \theta_l(k), \quad 1 \leq l \leq m_k,$$

*are mutually orthogonal, i.e.,*

$$\int_0^\pi \langle Y(x; \lambda_k) \theta_i(k), Y(x; \lambda_k) \theta_j(k) \rangle dx = 0, \quad \text{if } i \neq j.$$

**Proof.** Let  $v_1, \dots, v_{m_k}$  be a basis of  $\text{Null}(W(\lambda_k))$  and  $V_k = [v_1, \dots, v_{m_k}]$ . Then

$$V = \int_0^\pi V_k^* Y^*(x; \lambda_k) Y(x; \lambda_k) V_k dx$$

is an  $m_k \times m_k$  positive definite matrix. There exists an  $m_k \times m_k$  orthogonal matrix  $U$  which diagonalizes  $V$ , i.e.,  $U^* V U$  is a diagonal matrix. Let  $\theta_l(k)$ ,  $1 \leq l \leq m_k$ , denote the column vectors of  $V U$ . Then  $\theta_l(k)$  fulfills our requirement.  $\square$

According to **Lemma 2.1**, we define the following vector-valued function

$$(8) \quad \phi_l(x; \lambda, \lambda_k) = Y(x; \lambda) \theta_l(k), \quad 1 \leq l \leq m_k.$$

Then, the functions

$$\phi_l(x; \lambda_k, \lambda_k) = Y(x; \lambda_k) \theta_l(k), \quad 1 \leq l \leq m_k,$$

form an orthogonal basis of the eigenspace corresponding to the eigenvalue  $\lambda_k$ . From now on the eigenvalue problem  $(P, A, B, \mathcal{A}, \mathcal{B})$  shall be fixed. And, as it was mentioned in the introduction, for simplicity, we shall assume the completeness of the system of eigenfunctions  $\{\phi_l(x; \lambda_k, \lambda_k) : 1 \leq l \leq m_k, k = 0, 1, 2, \dots\}$  subject to the given boundary conditions.

Next we extend the idea used by Jodeit and Levitan in [3] to construct the kernel function for a related integral equation. We shall view an  $\mathbf{R}^N$ -vector  $\mathbf{v}$  as an  $N \times 1$  matrix. Choose  $c_k^i \in R$ ,  $1 \leq i \leq m_k, k = 0, 1, \dots$ , which converges so rapidly to zero that the matrix-valued function  $\mathcal{F}$ , defined by the following uniformly convergent series,

$$(9) \quad \mathcal{F}(x, y) = \sum_{k=0}^{\infty} \sum_{i=1}^{m_k} c_k^i \phi_i(x; \lambda_k, \lambda_k) \phi_i^*(y; \lambda_k, \lambda_k),$$

is continuous and has continuous first and second order derivatives. Then we construct the following integral equation :

$$(10) \quad K(x, y) + \mathcal{F}(x, y) + \int_0^x K(x, t) \mathcal{F}(t, y) dt = 0, \quad 0 \leq y < x \leq \pi.$$

We note that if we choose the sequence  $(c_k^i : 1 \leq i \leq m_k, k = 0, 1, 2, \dots)$  such that  $c_k^i = 0$  for all  $k \geq k_o$ ,  $i = 1, 2, \dots, m_k$ , where  $k_o$  is a fixed index, then the series in the right hand side of (9) is a finite series, and the equation (10) makes sense. This choice of  $(c_k^i)$  shall be used in section 4 to construct some concrete examples.

The existence of the solution of (10) can be easily proven by using iteration method. On the other hand, when we choose suitably those real numbers  $c_k^i, 1 \leq i \leq m_k, k \geq 0$ , we may prove the following uniqueness theorem.

**Theorem 2.2** *Suppose that the sequence  $(c_k^i)$  is chosen so that the series in (9) is uniformly convergent and has continuous first and second order derivatives, and*

$$(11) \quad 1 + c_k^i \|\phi_i(\cdot; \lambda_k, \lambda_k)\|^2 > 0, \quad 1 \leq i \leq m_k, \quad \forall k \geq 0.$$

*Then (10) has a unique solution for every  $x$ ,  $0 < x \leq \pi$ .*

**Proof.** The method for proving this theorem is similar to the one used for treating the scalar case in [3, **Theorem 1.1**]. It suffices to show that the only solution for the integral equation

$$(12) \quad \Delta(x, y) + \int_0^x \Delta(x, t) \mathcal{F}(t, y) dt = 0$$

is  $\Delta(x, y) \equiv 0$ , where  $\Delta(x, y)$  is the difference of two solution of (10). Denote  $\phi_{i,k}(x) = \phi_i(x; \lambda_k, \lambda_k)$  for convenience. Owing to the assumption about the completeness of eigenfunctions of  $(P, A, B, \mathcal{A}, \mathcal{B})$ , and the orthogonality (**Lemma 2.1**) of the eigenfunctions  $\phi_{i,k}(x)$ , we have

$$(13) \quad \Delta^*(x, y) = \sum_{k=0}^{\infty} \sum_{i=1}^{m_k} \frac{\phi_{i,k}(y)}{\|\phi_{i,k}\|} \left( \int_0^x \frac{\phi_{i,k}^*(t)}{\|\phi_{i,k}\|} \Delta^*(x, t) dt \right).$$

By (12), we have

$$\begin{aligned} \Delta^*(x, y) + \int_0^x \mathcal{F}^*(t, y) \Delta^*(x, t) dt &= 0, \\ (14) \quad \Delta(x, y) \Delta^*(x, y) + \int_0^x \Delta(x, y) \mathcal{F}^*(t, y) \Delta^*(x, t) dt &= 0. \end{aligned}$$

Integrating (14) with respect to  $y$ -variable from 0 to  $x$ , using (9) and (13), we obtain

$$(15) \quad \sum_{k=0}^{\infty} \sum_{i=0}^{m_k} \frac{1}{\|\phi_{i,k}\|} [1 + c_k^i \|\phi_{i,k}\|^2] \Omega_{i,k}(x) \Omega_{i,k}^*(x) = 0,$$

where

$$(16) \quad \Omega_{i,k}(x) = \int_0^x \Delta(x, t) \phi_{i,k}(t) dt.$$

Since  $\Omega_{i,k}(x) \Omega_{i,k}^*(x)$  is nonnegative definite, (15) and (11) imply that

$$\Omega_{i,k}(x) \Omega_{i,k}^*(x) = 0,$$

and hence  $\Omega_{i,k}(x) = 0$ , and by (16),

$$(17) \quad \int_0^x \Delta(x, t) \phi_{i,k}(t) dt = 0$$

for  $k = 0, 1, 2, \dots, i = 1, 2, \dots, m_k$ . Then by the completeness of eigenfunctions of  $(P, A, B, \mathcal{A}, \mathcal{B})$ , (17) implies  $\Delta(x, y) = 0$ . This completes the proof.  $\square$

Now we face the question : “ Does the matrix-valued function  $K(x, y)$  determined by the above theorem also satisfy some wave equation with which we are familiar as in the scalar case ? ” The answer is affirmative, as shown below.

**Theorem 2.3** *Assumption as Theorem 2.2. The solution  $K(x, y)$  of (10) satisfies the following partial differential equation*

$$(18) \quad \frac{\partial^2}{\partial x^2} K - Q(x)K = \frac{\partial^2}{\partial y^2} K - KP(y),$$

where  $Q(x) = P(x) + 2d/dx K(x, x)$ , and it also satisfies the following conditions:

$$\begin{aligned} (19) \quad K(x, y) &= 0, \quad \text{if } y > x, \\ K(x, 0)A^* + \frac{\partial}{\partial y} K|_{y=0} B^* &= 0, \end{aligned}$$

$$(20) \quad K(x, x) = \frac{1}{2} \int_0^x [Q(t) - P(t)] dt - \mathcal{F}(0, 0),$$

where

$$(21) \quad \mathcal{F}(0, 0) = B^* \left( \sum_{k=0}^{\infty} \sum_{i=1}^{m_k} c_k^i \theta_i(k) \theta_i^*(k) \right) B.$$

**Proof.** Denote

$$\mathcal{J}(x, y) = K(x, y) + \mathcal{F}(x, y) + \int_0^x K(x, t) \mathcal{F}(t, y) dt.$$

Then by (10),  $\mathcal{J} = 0$ , hence  $\mathcal{J}_{xx} = \mathcal{J}_{yy} = 0$ . On the other hand, as

$$\begin{aligned} \mathcal{J}_{xx} &= \frac{\partial^2}{\partial x^2} K + [P(x) + (\frac{d}{dx} K(x, x) + \frac{\partial}{\partial x} K(x, x))] \mathcal{F}(x, y) \\ &\quad - \sum_{k=0}^{\infty} \sum_{i=1}^{m_k} \lambda_k c_k^i \phi_i(x; \lambda_k, \lambda_k) \phi_i^*(x; \lambda_k, \lambda_k) \\ &\quad + K(x, x) \frac{\partial}{\partial x} \mathcal{F}(x, y) + \int_0^x \frac{\partial^2}{\partial x^2} K(x, t) \mathcal{F}(t, y) dt, \end{aligned}$$

$$\begin{aligned} \mathcal{J}_{yy} &= \frac{\partial^2}{\partial y^2} K + (\mathcal{F}(x, y) + \int_0^x K(x, t) \mathcal{F}(t, y) dt) P(y) \\ &\quad + \sum_{k=0}^{\infty} \sum_{i=1}^{m_k} \lambda_k c_k^i \phi_i(x; \lambda_k, \lambda_k) \phi_i^*(y; \lambda_k, \lambda_k) \\ &\quad - \int_0^x K(x, t) [\sum_{k=0}^{\infty} \sum_{i=1}^{m_k} \lambda_k c_k^i \phi_i(t; \lambda_k, \lambda_k) \phi_i^*(y; \lambda_k, \lambda_k)] dt, \end{aligned}$$

we have

$$\begin{aligned} 0 &= \mathcal{J}_{xx} - \mathcal{J}_{yy} + \mathcal{J}P(y) \\ &= \frac{\partial^2}{\partial x^2} K - \frac{\partial^2}{\partial y^2} K + [P(x) + 2\frac{d}{dx} K(x, x) - \frac{\partial}{\partial y} K|_{y=x}] \mathcal{F}(x, y) \\ &\quad + K(x, x) \frac{\partial}{\partial x} \mathcal{F} + \int_0^x \frac{\partial^2}{\partial x^2} K(x, t) \mathcal{F}(t, y) dt \\ &\quad + \int_0^x K(x, t) [\sum_{k=0}^{\infty} \sum_{i=1}^{m_k} c_k^i (\lambda_k \phi_i(t; \lambda_k, \lambda_k)) \phi_i^*(y; \lambda_k, \lambda_k)] dt. \end{aligned}$$

Replacing  $\lambda_k \phi_i(t; \lambda_k, \lambda_k)$  by  $-\phi_i''(t; \lambda_k, \lambda_k) + P(t) \phi_i(t; \lambda_k, \lambda_k)$  in the last integral and using integration by parts twice, we obtain

$$\begin{aligned} 0 &= \mathcal{J}_{xx} - \mathcal{J}_{yy} + \mathcal{J}P(y) \\ &= \frac{\partial^2}{\partial x^2} K - \frac{\partial^2}{\partial y^2} K + (P(x) + 2\frac{d}{dx}) \mathcal{F}(x, y) \\ &\quad + [K(x, 0) \frac{\partial}{\partial x} \mathcal{F}(x, y)|_{x=0} - \frac{\partial}{\partial y} K(x, y)|_{y=0} \mathcal{F}(0, y)] \\ &\quad + \int_0^x [\frac{\partial^2}{\partial x^2} K(x, t) - \frac{\partial^2}{\partial t^2} K(x, t) + K(x, t) P(t)] \mathcal{F}(t, y) dt. \end{aligned}$$

Finally, let  $Q(x) = P(x) + 2d/dxK(x, x)$ , we have

$$\begin{aligned}
(22) \quad 0 &= \mathcal{J}_{xx} - \mathcal{J}_{yy} + \mathcal{J}P(y) - Q(x)\mathcal{J} \\
&= + [K(x, 0)\frac{\partial}{\partial x}\mathcal{F}(x, y)|_{x=0} - \frac{\partial}{\partial y}K(x, y)|_{y=0}\mathcal{F}(0, y)] \\
&\quad + [\frac{\partial^2}{\partial x^2}K - \frac{\partial^2}{\partial y^2}K - Q(x)K + KP(y)] \\
&\quad + \int_0^x [\frac{\partial^2}{\partial x^2}K - \frac{\partial^2}{\partial t^2}K - Q(x)K + KP(t)]\mathcal{F}(t, y)dt,
\end{aligned}$$

where the function

$$K(x, 0)\frac{\partial}{\partial x}\mathcal{F}(x, y)|_{x=0} - \frac{\partial}{\partial y}K(x, y)|_{y=0}\mathcal{F}(0, y)$$

vanishes if and only if (19) holds. Then (22) becomes an integral equation of the same type as (12). Hence we have (18).

By (9) and (10), we see that

$$K(0, 0) = -\mathcal{F}(0, 0) = B^*(\sum_{k=0}^{\infty} \sum_{i=1}^{m_k} c_k^i \theta_i(k) \theta_i^*(k))B.$$

(20) is a consequence of the fundamental theorem of calculus from the definition of  $Q(x)$  given above.  $\square$

**Theorem 2.4** *If  $K(x, y)$  is determined by Theorem 2.3, then, for every complex  $\lambda$ ,  $k \geq 0$  and  $1 \leq l \leq m_k$ , the vector-valued function  $\psi_l(x; \lambda, \lambda_k)$  defined by*

$$(23) \quad \psi_l(x; \lambda, \lambda_k) = \phi_l(x; \lambda, \lambda_k) + \int_0^x K(x, t)\phi_l(t; \lambda, \lambda_k)dt$$

*is a solution of the vectorial differential equation*

$$(24) \quad -\psi'' + Q(x)\psi = \lambda\psi, \quad 0 \leq x \leq \pi,$$

*where  $Q(x) = P(x) + 2d/dxK(x, x)$ , and  $\psi_1(x; \lambda_k, \lambda_k), \dots, \psi_{m_k}(x; \lambda_k, \lambda_k)$  are linearly independent. In addition, it also satisfies the following initial conditions:*

$$(25) \quad \psi_l(0; \lambda, \lambda_k) = B^*\theta_l(k),$$

$$\begin{aligned}
(26) \quad \psi_l'(0; \lambda, \lambda_k) &= (-A^* + K(0, 0)B^*)\theta_l(k) \\
&= (-A^* - B^*(\sum_{r=0}^{\infty} \sum_{i=1}^{m_r} c_r^i \theta_i(r) \theta_i^*(r))BB^*)\theta_l(k),
\end{aligned}$$

*or, equivalently,*

$$(27) \quad B\psi_l'(0; \lambda, \lambda_k) + \tilde{A}\psi_l(0; \lambda, \lambda_k) = \mathbf{0},$$

*where*

$$(28) \quad \tilde{A} = A - BK(0, 0).$$



**Proof.** By (23), we have

$$\begin{aligned}
\psi_l''(x; \lambda, \lambda_k) &= \phi_l''(x; \lambda, \lambda_k) + \left[ \int_0^x K(x, t) \phi_l(t; \lambda, \lambda_k) dt \right]'' \\
(29) \quad &= (P(x) - \lambda I) \phi_l(x; \lambda, \lambda_k) + \int_0^x \frac{\partial^2}{\partial x^2} K(x, t) \phi_l(t; \lambda, \lambda_k) dt \\
&\quad + K(x, x) \phi_l'(x; \lambda, \lambda_k) + \frac{\partial}{\partial x} K(x, t)|_{t=x} \phi_l(x; \lambda, \lambda_k),
\end{aligned}$$

and

$$\begin{aligned}
\lambda \psi_l(x; \lambda, \lambda_k) &= \lambda \phi_l(x; \lambda, \lambda_k) + \int_0^x K(x, t) \lambda \phi_l(t; \lambda, \lambda_k) dt \\
&= \lambda \phi_l(x; \lambda, \lambda_k) + \int_0^x K(x, t) \lambda Y(t; \lambda) \theta_i(k) dt \\
&= \lambda \phi_l(x; \lambda, \lambda_k) + \int_0^x K(x, t) [-Y''(t; \lambda) + P(t) Y(t; \lambda)] \theta_i(k) dt \\
&= \lambda \phi_l(x; \lambda, \lambda_k) + \int_0^x K(x, t) P(t) \phi_l(t; \lambda, \lambda_k) dt \\
&\quad - \int_0^x K(x, t) Y''(t; \lambda) \theta_i(k) dt.
\end{aligned}$$

Using integration by parts twice on the last integral, we have

$$\begin{aligned}
\lambda \psi_l(x; \lambda, \lambda_k) &= \lambda \phi_l(x; \lambda, \lambda_k) + \int_0^x K(x, t) P(t) \phi_l(t; \lambda, \lambda_k) dt \\
(30) \quad &\quad + (-K(x, 0) A^* - \frac{\partial}{\partial y} K|_{y=0} B^*) \theta_l(k) \\
&\quad - K(x, x) \phi_l'(x; \lambda, \lambda_k) + \frac{\partial}{\partial y} K|_{y=x} \phi_l(x; \lambda, \lambda_k) \\
&\quad - \int_0^x \frac{\partial^2}{\partial t^2} K(x, t) \phi_l(t; \lambda, \lambda_k) dt.
\end{aligned}$$

Then, by **Theorem 2.3**, we have

$$\begin{aligned}
&-\psi_l''(x; \lambda, \lambda_k) + (\lambda I - Q(x)) \psi_l(x; \lambda, \lambda_k) \\
&= (-K(x, 0) A^* - \frac{\partial}{\partial y} K|_{y=0} B^*) \theta_l(k) \\
&\quad + \int_0^x \left[ \frac{\partial^2}{\partial x^2} K - \frac{\partial^2}{\partial t^2} K - Q(x) K + K P(t) \right] dt \\
&= \mathbf{0}.
\end{aligned}$$

Besides, for  $1 \leq l \leq m_k$ ,

$$\psi_l(0; \lambda, \lambda_k) = \phi_l(0; \lambda, \lambda_k) = B^* \theta_l(k),$$

$$\begin{aligned}
\psi'_l(0; \lambda, \lambda_k) &= \phi'_l(0; \lambda, \lambda_k) + K(0, 0)\phi_l(0; \lambda, \lambda_k) \\
&= (-A^* + K(0, 0)B^*)\theta_l(k) \\
&= (-A^* - B^*(\sum_{r=0}^{\infty} \sum_{i=1}^{m_r} c_r^i \theta_i(r) \theta_i^*(r))BB^*)\theta_l(k).
\end{aligned}$$

If we denote  $\tilde{A} = A - BK(0, 0)$ , then  $B\tilde{A}^* = \tilde{A}B^*$ , and, by using  $BA^* = AB^*$ , we have

$$\begin{aligned}
&B\psi'_l(0; \lambda, \lambda_k) + \tilde{A}\psi_l(0; \lambda, \lambda_k) \\
&= B(-A^* + K(0, 0)B^*)\theta_l(k) + (A - BK(0, 0))B^*\theta_l(k) \\
&= \mathbf{0}.
\end{aligned}$$

The linear independence of those vector-valued functions  $\psi_l(x; \lambda_k, \lambda_k)$ ,  $1 \leq l \leq m_k$  can be proven by using (23), Gronwall's lemma and the linear independence of those functions  $\phi_l(x; \lambda_k, \lambda_k)$ ,  $1 \leq l \leq m_k$ .  $\square$

At the end of this section, we state a theorem which indicates the possible candidates for eigenfunctions of the isospectral problem  $(Q, \tilde{A}, B, \tilde{A}, \mathcal{B})$  of  $(P, A, B, \mathcal{A}, \mathcal{B})$  which shall be described in next section. Furthermore, we may also use this theorem to construct that an isospectral Dirichlet's problem of a given Dirichlet's problem.

**Theorem 2.5** *Suppose  $\lambda_k \in \sigma(P, A, B, \mathcal{A}, \mathcal{B})$ ,  $k \geq 0$ . Then*

$$\begin{aligned}
(31) \quad \psi_l(x; \lambda_k, \lambda_k) &= \phi_l(x; \lambda_k, \lambda_k) \\
&\quad - \sum_{r=0}^{\infty} \sum_{i=1}^{m_r} c_r^i \psi_i(x; \lambda_r, \lambda_r) \int_0^x \phi_i^*(t; \lambda_r, \lambda_r) \phi_l(t; \lambda_k, \lambda_k) dt
\end{aligned}$$

for all  $1 \leq l \leq m_k$ .

**Proof.** The proof is similar to the one of [3, **Theorem 1.4**]. Denote  $\phi_{i,k}(x) = \phi_i(x; \lambda_k, \lambda_k)$ . By (9), and the integral equation (10), we have

$$\begin{aligned}
K(x, y) &= -\mathcal{F}(x, y) - \int_0^x K(x, t)\mathcal{F}(t, y)dt \\
&= -\sum_{r=0}^{\infty} \sum_{i=1}^{m_r} c_r^i [\phi_{i,r}(x) + \int_0^x K(x, t)\phi_{i,r}(t)dt] \phi_{i,r}^*(y).
\end{aligned}$$

Then, by (23), we have

$$(32) \quad K(x, t) = -\sum_{r=0}^{\infty} \sum_{i=1}^{m_r} c_r^i \psi_i(x; \lambda_r, \lambda_r) \phi_i^T(t; \lambda_r, \lambda_r).$$

Apply (32), (23) implies (31).  $\square$

### 3 Isospectral problem

Those theorems in previous section enable us to construct an isospectral problem from a given eigenvalue problem  $(P, A, B, \mathcal{A}, \mathcal{B})$  and a sequence of real numbers  $c_k^i, 1 \leq i \leq m_k, k \geq 0$ , where the sequence  $(c_k^i)$  satisfies the assumption of **Theorem 2.2**. As **Theorem 2.4** states, for any  $k \geq 0$ , and for each  $l, 1 \leq l \leq m_k$ , the vector-valued function  $\psi_l(x; \lambda_k, \lambda_k)$  satisfies the boundary condition

$$B\psi'(0) + \tilde{A}\psi(0) = \mathbf{0},$$

where  $\tilde{A}$  is given by (28). Hence, the final step for constructing isospectral problem is to determine the form of boundary condition to be satisfied at  $x = \pi$ . For this purpose we use formula (31). By (31), we have

$$(33) \quad \psi_l(\pi; \lambda_k, \lambda_k) = \frac{\phi_l(\pi; \lambda_k, \lambda_k)}{1 + c_k^l \|\phi_l(x; \lambda_k, \lambda_k)\|^2}.$$

Differentiating (31) with respect to  $x$  and evaluating it at  $\pi$ , we have

$$\begin{aligned} \psi_l'(\pi; \lambda_k, \lambda_k) &= \phi_l'(\pi; \lambda_k, \lambda_k) - c_k^l \psi_l'(\pi; \lambda_k, \lambda_k) \|\phi_l(x; \lambda_k, \lambda_k)\|^2 \\ &\quad - \left[ \sum_{r=0}^{\infty} \sum_{i=1}^{m_r} c_r^i \psi_i(\pi; \lambda_r, \lambda_r) \phi_l^*(\pi; \lambda_r, \lambda_r) \right] \phi_l(\pi; \lambda_k, \lambda_k), \end{aligned}$$

and, hence we have

$$\begin{aligned} &(1 + c_k^l \|\phi_l(x; \lambda_k, \lambda_k)\|^2) \psi_l'(\pi; \lambda_k, \lambda_k) \\ &= \phi_l'(\pi; \lambda_k, \lambda_k) - \left[ \sum_{r=0}^{\infty} \sum_{i=1}^{m_r} \frac{c_r^i \phi_i(\pi; \lambda_r, \lambda_r) \phi_i^*(\pi; \lambda_r, \lambda_r)}{1 + c_r^i \|\phi_i(x; \lambda_r, \lambda_r)\|^2} \right] \phi_l(\pi; \lambda_k, \lambda_k) \end{aligned}$$

Acting on the above identity by  $\mathcal{B}$  and using the condition  $\mathcal{B}\phi_l'(\pi; \lambda_k, \lambda_k) + \mathcal{A}\phi_l(\pi; \lambda_k, \lambda_k) = 0$ , we have

$$\begin{aligned} &\mathcal{B}(1 + c_k^l \|\phi_l(x; \lambda_k, \lambda_k)\|^2) \psi_l'(\pi; \lambda_k, \lambda_k) \\ &= \mathcal{B}\phi_l'(\pi; \lambda_k, \lambda_k) - \mathcal{B} \left[ \sum_{r=0}^{\infty} \sum_{i=1}^{m_r} \frac{c_r^i \phi_i(\pi; \lambda_r, \lambda_r) \phi_i^*(\pi; \lambda_r, \lambda_r)}{1 + c_r^i \|\phi_i(x; \lambda_r, \lambda_r)\|^2} \right] \phi_l(\pi; \lambda_k, \lambda_k) \\ &= -(\mathcal{A} + \mathcal{B} \left[ \sum_{r=0}^{\infty} \sum_{i=1}^{m_r} \frac{c_r^i \phi_i(\pi; \lambda_r, \lambda_r) \phi_i^*(\pi; \lambda_r, \lambda_r)}{1 + c_r^i \|\phi_i(x; \lambda_r, \lambda_r)\|^2} \right]) \phi_l(\pi; \lambda_k, \lambda_k). \end{aligned}$$

Then, by (33), we have

$$\mathcal{B}\psi_l'(\pi; \lambda_k, \lambda_k) = -(\mathcal{A} + \mathcal{B} \left[ \sum_{r=0}^{\infty} \sum_{i=1}^{m_r} \frac{c_r^i \phi_i(\pi; \lambda_r, \lambda_r) \phi_i^*(\pi; \lambda_r, \lambda_r)}{1 + c_r^i \|\phi_i(x; \lambda_r, \lambda_r)\|^2} \right]) \psi_l(\pi; \lambda_k, \lambda_k),$$

and hence,

$$\mathcal{B}\psi'_l(\pi; \lambda_k, \lambda_k) + \tilde{\mathcal{A}}\psi_l(\pi; \lambda_k, \lambda_k) = \mathbf{0},$$

where

$$(34) \quad \tilde{\mathcal{A}} = \mathcal{A} + \mathcal{B} \left[ \sum_{r=0}^{\infty} \sum_{i=1}^{m_r} \frac{c_r^i \phi_i(\pi; \lambda_r, \lambda_r) \phi_i^*(\pi; \lambda_r, \lambda_r)}{1 + c_r^i \|\phi_i(x; \lambda_r, \lambda_r)\|^2} \right].$$

In fact, by (32), (34) can be simplified as

$$(35) \quad \tilde{\mathcal{A}} = \mathcal{A} - \mathcal{B}K(\pi, \pi).$$

Furthermore, if we can prove that, for any  $\mathbf{R}^N$ -valued function  $f$  satisfying  $Bf'(0) + \tilde{\mathcal{A}}f(0) = \mathbf{0}$ ,  $\mathcal{B}f'(\pi) + \tilde{\mathcal{A}}f(\pi) = \mathbf{0}$ , and

$$\int_0^\pi \langle f(x), \psi_l(x; \lambda_k, \lambda_k) \rangle dx = 0, \quad k \geq 0, \quad l = 1, \dots, m_k,$$

we have  $f \equiv \mathbf{0}$ , then the set  $\{\psi_l(x; \lambda_k, \lambda_k) : k \geq 0, 1 \leq l \leq m_k\}$  is complete, and, by **Theorem 2.4**, we have  $\Sigma(P, A, B, \mathcal{A}, \mathcal{B}) = \Sigma(Q, \tilde{\mathcal{A}}, B, \tilde{\mathcal{A}}, \mathcal{B})$ . For our purpose, let

$$T(f)(x) = \int_0^x K(x, t)f(t)dt.$$

Then

$$T^*(g)(t) = \int_t^\pi K^*(x, t)g(x)dx.$$

Writing  $\psi_l(x; \lambda_k, \lambda_k) = (I + T)\phi_l(x; \lambda_k, \lambda_k)$ ,  $1 \leq l \leq m_k$ , we have

$$\begin{aligned} 0 &= \int_0^\pi \langle f(x), \psi_l(x; \lambda_k, \lambda_k) \rangle dx = \int_0^\pi \langle f(x), (I + T)\phi_l(x; \lambda_k, \lambda_k) \rangle dx \\ &= \int_0^\pi \langle (I + T^*)f(x), \phi_l(x; \lambda_k, \lambda_k) \rangle dx. \end{aligned}$$

Now set  $g = (I + T^*)f$ . If we show that  $Bg'(0) + Ag(0) = \mathbf{0}$ , and  $\mathcal{B}g'(\pi) + \mathcal{A}g(\pi) = \mathbf{0}$ , then by the completeness of the set  $\{\phi_l(x; \lambda_k, \lambda_k) : k \geq 0, 1 \leq l \leq m_k\}$ , we have  $g \equiv \mathbf{0}$  and hence  $f \equiv \mathbf{0}$ . We only check the identity  $Bg'(0) + Ag(0) = \mathbf{0}$ , the other part can be proved by similar argument. Suppose  $Bf'(0) + \tilde{\mathcal{A}}f(0) = \mathbf{0}$ . Then, using (19) and (28), we have

$$\begin{aligned} Bg'(0) + Ag(0) &= B[f'(0) - K^*(0, 0)f(0) \\ &\quad + \int_0^\pi K_t^*(x, 0)f(x)dx] + A[f(0) + \int_0^\pi K^*(x, 0)f(x)dx] \\ &= Bf'(0) - BK^*(0, 0)f(0) \\ &\quad + Af(0) + \int_0^\pi [BK_t^*(x, 0) + AK^*(x, 0)]f(x)dx \\ &= Bf'(0) + [A - BK(0, 0)]f(0) = Bf'(0) + \tilde{\mathcal{A}}f(0) = \mathbf{0}. \end{aligned}$$

As a conclusion of the previous arguments, we have the following theorem.

**Theorem 3.1** *Let  $m_k$  denote the multiplicity of  $\lambda_k$  in  $\sigma(P, A, B, \mathcal{A}, \mathcal{B})$ . Suppose  $\{c_k^i, 1 \leq i \leq m_k, k \geq 0\}$  is a sequence, satisfying the condition (11) and making  $\mathcal{F}(x, y)$  in (9) a  $C^2$ -function,  $Q(x)$  is as that defined in **Theorem 2.3**,  $\tilde{A}$  and  $\tilde{\mathcal{A}}$  are as those defined in (28) and (35). Then  $\Sigma(P, A, B, \mathcal{A}, \mathcal{B}) = \Sigma(Q, \tilde{A}, B, \tilde{\mathcal{A}}, \mathcal{B})$ .*

As a final remark, we note that if  $A = I, B = 0, \mathcal{A} = I$ , and  $\mathcal{B} = 0$  in (1), then the matrices  $\tilde{A}$  and  $\tilde{\mathcal{A}}$  in **Theorem 3.1** are equal to  $I$ , the identity matrix. Hence, for a given Dirichlet problem  $(P, I, 0, I, 0)$ , the isospectral problem constructed in **Theorem 3.1** is also a Dirichlet problem.

## 4 Examples

In this section, we use our theory to construct some examples which have some significant meaning the scalar case can not tell.

Suppose  $\lambda_o$  is an eigenvalue of (1) with multiplicity  $m_o$ . Let  $\phi_o(x) = \text{col}(\phi_1(x), \phi_2(x), \dots, \phi_N(x))$  be an eigenfunction corresponding to  $\lambda_o$ . Take

$$\begin{aligned} \mathcal{F}(x, y) &= c\phi_o(x)\phi_o^*(y) \\ &= c \begin{pmatrix} \phi_1(x)\phi_1(y) & \phi_1(x)\phi_2(y) & \cdots & \phi_1(x)\phi_N(y) \\ \phi_2(x)\phi_1(y) & \phi_2(x)\phi_2(y) & \cdots & \phi_2(x)\phi_N(y) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N(x)\phi_1(y) & \phi_N(x)\phi_2(y) & \cdots & \phi_N(x)\phi_N(y) \end{pmatrix}. \end{aligned}$$

Plugging it into (10), and letting  $k_{ij}(x, y)$  denote the  $(i, j)$  entry of  $K(x, y)$ , we have

$$(36) \quad k_{ij}(x, y) + c\phi_i(x)\phi_j(y) + (c \int_0^x (\sum_{r=1}^N k_{ir}(x, t)\phi_r(t))dt)\phi_j(y) = 0,$$

for  $i = 1, \dots, N$  and  $j = 1, \dots, N$ .

We shall show that

$$(37) \quad k_{ij}(x, y) = -\frac{c\phi_i(x)\phi_j(y)}{1 + c \int_0^x |\phi_o(t)|^2 dt}.$$

For  $i$  fixed, consider the equations

$$(38) \quad k_{ij}(x, y) + c\phi_i(x)\phi_j(y) + (c \int_0^x (\sum_{r=1}^N k_{ir}(x, t)\phi_r(t))dt)\phi_j(y) = 0,$$

$1 \leq j \leq N$ . Multiplying the  $j$ th equation by  $\phi_j(y)$ , integrating it from 0 to  $x$  with respect to  $y$ , and denoting  $p_{i,j}(x) = \int_0^x k_{ij}(x, t)\phi_j(t)dt$  and  $\alpha_j(x) = \int_0^x \phi_j^2(t)dt$ ,

$1 \leq j \leq N$ , we have the following linear system of equations with unknowns  $p_{ij}(x)$ ,

$$(39) \quad p_{ij}(x) + c\phi_i(x)\alpha_j(x) + c\alpha_j(x)\left(\sum_{r=1}^N p_{ir}(x)\right) = 0, \quad 1 \leq j \leq N.$$

Solving this system, we obtain

$$(40) \quad p_{ij}(x) = -\frac{c\phi_i(x)\alpha_j(x)}{1 + c\int_0^x |\phi_\circ(t)|^2 dt}, \quad 1 \leq j \leq N.$$

Note that, by (38),

$$k_{ij}(x, y) = -c\phi_i(x)\phi_j(y) - (c\int_0^x (\sum_{r=1}^N k_{ir}(x, t)\phi_r(t))dt)\phi_j(y).$$

Hence if we plug (40) into (38), we obtain (37). It also follows from (37), that

$$K(x, y) = -\frac{1}{1 + c\int_0^x |\phi_\circ(t)|^2 dt} \mathcal{F}(x, y).$$

Hence, according to **Theorem 3.1**, by setting

$$(41) \quad \begin{aligned} Q(x) &= P(x) - 2\frac{d}{dx}\left[\frac{\phi_\circ(x)\phi_\circ^*(x)}{1 + c\int_0^x |\phi_\circ(t)|^2 dt}\right], \\ \tilde{A} &= A - cB\phi_\circ(0)\phi_\circ^*(0), \\ \tilde{\mathcal{A}} &= \mathcal{A} - \frac{c\phi_\circ(\pi)\phi_\circ^*(\pi)}{1 + c\|\phi_\circ\|^2}, \end{aligned}$$

we have  $\Sigma(P, A, B, \mathcal{A}, \mathcal{B}) = \Sigma(Q, \tilde{A}, B, \tilde{\mathcal{A}}, \mathcal{B})$ .

As an example to the above construction, we construct the following eigenvalue problem which has an eigenvalue of multiplicity 2.

Let  $I$  be the  $2 \times 2$  identity matrix. Take

$$P(x) = \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}, \quad A = \mathcal{A} = I, \quad B = \mathcal{B} = 0.$$

Then one can verify that for the eigenvalue problem  $(P, I, 0, I, 0)$ , 1 is an eigenvalue of multiplicity 2, and the other eigenvalues are all simple. The eigenspace corresponding to 1 is the vector space spanned by the two vector-valued functions  $(\sin(2x), 0)^*$  and  $(0, \sin(x))^*$ . Choosing  $(\sin(2x), \sin(x))^*$  as the eigenfunction which plays the role of  $\phi_\circ(x)$  in the above construction,  $c = 1$ , and using

(41), we have

$$Q(x) = \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix} - \frac{d}{dx} \left[ \frac{2}{1 + \int_0^x (\sin^2(t) + \sin^2(2t)) dt} \cdot \begin{pmatrix} \sin^2(2x) & \sin(2x) \sin(x) \\ \sin(x) \sin(2x) & \sin^2(x) \end{pmatrix} \right].$$

and  $\tilde{A} = \tilde{\mathcal{A}} = I$ . Note that the matrix potential function  $Q(x)$  is not simultaneously diagonalizable since, as checked by computation, the matrix-valued functions  $Q(x)$  and  $Q'(x)$  do not commute. On the other hand, if we take  $(0, \sin(x))^*$  instead, then we find that  $Q(x)$  is a diagonal matrix-valued function and is of the following form

$$Q(x) = \begin{pmatrix} -3 & 0 \\ 0 & \frac{d}{dx} \left( \frac{-2 \sin^2(x)}{1 + \int_0^x \sin^2(t) dt} \right) \end{pmatrix}.$$

There are lots of interesting phenomena can be observed from our construction, which shall be observed later.

**Acknowledgements.** (i) The author show his gratitude to his Ph D. adviser Professor C. L. Shen for his instruction. (ii) The author became aware that Professor B. M. Levitan and Max. Jodeit also obtained analogous results in [4].

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